ANALYSIS OF THE FLOW OF AN IRREGULAR VISCOUS FLUID BETWEEN TWO REVOLVING CYLINDERS

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We propose a mathematical model for the flow of an irregular viscous fluid between two revolving cylinders in a bipolar coordinate system. We present an analytic solution of the system of equations obtained, and we also obtain formulas for determining velocity fields, temperature fields, and energy characteristics.

A considerable number of native and foreign works [1-6] are devoted to the theoretical and experimental study of the flow of an irregular viscous fluid between two revolving cylinders, applicable to the processes of rolling polymers and plastics. In most cases the flow parameters are calculated in an isothermic approximation [1, 6]. Existing calculation methods, which take heat processes in the flow into account, study only a small part of the deformation region (section A, Fig. 1), in which the flow is assumed to be one-dimensional. It is of great practical interest to analyze the process under the condition when the magnitudes of the outlet coordinate are large (Fig. 1). In this case we must determine all the velocity components. An analytic solution of this problem is given in the present study.

The steady flow of a viscous incompressible fluid between two revolving cylinders can be described by a system of equations that contains an equation of continuity, an equation of the conservation of momentum, and an energy equation [2]:

$$\nabla_i v^i = 0, \tag{1}$$

$$\rho v^j \nabla_j v^i = g^{ij}\nabla_j P + \nabla_j \tau^{ij}, \tag{2}$$

$$\lambda\Delta T - \rho c_v v^i \nabla_i T + \tau^{ij} e_{ij} = 0. \tag{3}$$

We assume that the properties of the material itself are described by the rheological equation of Ostwald de Waele:

$$\tau^{ij} = \mu(T)\left|\frac{1}{2}J_2\right|^{\frac{n-1}{2}} e^{ij}. \tag{4}$$

The components of the metric tensor are

$$g_{\alpha\beta} = g_{\beta\alpha} = 0, \quad g_{\alpha\alpha} = g_{\beta\beta} = \frac{a^2}{(\operatorname{ch}\alpha - \cos\beta)^2} = h^2,$$

i.e., the coordinate system is orthogonal.

We can write system (1)-(4) in a bipolar coordinate system, since the distances between the poles of the roller surface coincide with the coordinates $\pm\alpha_0$, which considerably simplifies the boundary conditions. In addition, the coordinate grid of the bipolar coordinate system is close to the flow line, so, in expanding the physical components of the velocity u and v in powers of the small parameter ε (which will be further determined), we can assume that one component is ε times less than the other:

$$\frac{\partial}{\partial\beta}(hu) + \frac{\partial}{\partial\alpha}(hv) = 0, \tag{5}$$

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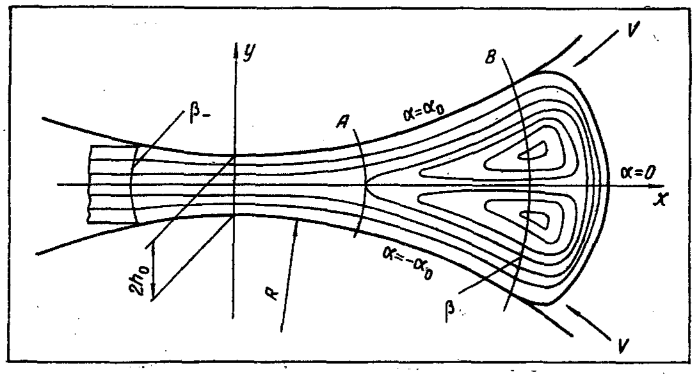

Fig. 1. Scheme of the deformation region.

$$\rho\left\{(uh)\frac{\partial}{\partial\beta}\left(\frac{u}{h}\right)+(vh)\frac{\partial}{\partial\alpha}\left(\frac{u}{h}\right)+h\frac{\partial h}{\partial\beta}\left[\left(\frac{u}{h}\right)^2-\left(\frac{v}{h}\right)^2\right]+2h\frac{\partial h}{\partial\beta}uv\right\}=$$

$$=-\frac{\partial P}{\partial\beta}+\frac{\partial}{\partial\alpha}\tau_\beta^\alpha+\frac{\partial}{\partial\beta}\tau_\alpha^\beta+2\tau_\beta^\alpha\frac{1}{h}\cdot\frac{\partial h}{\partial\alpha}+(\tau_\beta^\beta-\tau_\alpha^\alpha)\frac{1}{h}\cdot\frac{\partial h}{\partial\beta}, \tag{6}$$

$$\frac{\lambda}{h^2}\left(\frac{\partial^2 T}{\partial\alpha^2}+\frac{\partial^2 T}{\partial\beta^2}\right)-\frac{\rho c_v}{h}\left(u\frac{\partial T}{\partial\beta}+v\frac{\partial T}{\partial\alpha}\right)+\tau^{ij}e_{ij}=0, \tag{7}$$

and the components of tensor e_j^i are

$$e_\beta^\beta=2\left(\frac{1}{h}\cdot\frac{\partial u}{\partial\beta}+\frac{v}{h^2}\cdot\frac{\partial h}{\partial\alpha}\right),\quad e_\alpha^\beta=e_\beta^\alpha=\frac{\partial}{\partial\alpha}\left(\frac{u}{h}\right)+\frac{\partial}{\partial\beta}\left(\frac{v}{h}\right). \tag{8}$$

The projection of Eq. (2) onto the α axis and the component of tensor e_α^α are obtained by the interchange of indices α and β and the simultaneous replacement of the velocity component u by v and vice versa. The viscosity of most commercial thermoplastics allows us to disregard the inertial terms in the equation of motion [4]. A characteristic of the problem given is that the distance between the surfaces of the cylinders is much less than the length of the deformation region. Thus, because of the small parameter* we can assume that $\varepsilon = \alpha_0/\beta'$. We introduce the new variables $\xi = \beta/\beta'$; $\eta = \alpha/\alpha_0$ and expand both velocity components in the small parameter ε,

$$u=V\left(u_0(\xi,\ \eta)+\varepsilon u_1(\xi,\ \eta)+\ldots\right), \tag{9}$$

$$v=V\left(\varepsilon v_1(\xi,\ \eta)+\ldots\right). \tag{10}$$

After substituting (9) and (10) into the system of equations (5)-(7) and disregarding terms of order ε^2, we obtain the system of equations

$$\frac{\partial}{\partial\beta}(uh)+\frac{\partial}{\partial\alpha}(vh)=0, \tag{11}$$

$$\frac{\partial}{\partial\beta}P=\frac{\partial}{\partial\alpha}\left\{\mu\left|\frac{\partial}{\partial\alpha}\left(\frac{u}{h}\right)\right|^n\operatorname{sign}\left[\frac{\partial}{\partial\alpha}\left(\frac{u}{h}\right)\right]\right\};\quad \frac{\partial}{\partial\alpha}P=0, \tag{12}$$

$$\frac{\lambda}{h^2}T''_{\alpha\alpha}-\frac{\rho c_v}{h}(uT'_\beta+vT'_\alpha)+\mu\left|\frac{\partial}{\partial\alpha}\left(\frac{u}{h}\right)\right|^{n+1}=0. \tag{13}$$

We note that system (11)-(13) transforms to a system of equations in a Cartesian coordinate system, with the simplifications of boundary-layer theory taken into account [2] if we set h = 1. The boundary conditions for the equal velocity of the cylinders of single radius are written as follows:

1) $P=0,\quad T=\begin{cases} T_1 & \alpha=\alpha_0, \\ T_0 & \alpha\in(-\alpha_0,\ \alpha_0) \\ T_2 & \alpha=-\alpha_0; \end{cases}$ for $\beta=\beta_+$,

2) to find outlet coordinate β we assume that at the outlet of the deformation region the pressure becomes zero $-P = 0$ for $\beta = \beta_-$, $\partial P/\partial\beta = 0$;

3) $u=V$ for $\alpha=\pm\alpha_0$;

4) $\frac{\partial}{\partial\alpha}\left(\frac{u}{h}\right)=v=0$ for $\alpha=0$.

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*For the case of rollers used in industry, $\alpha_0 \simeq 0.1-0.05$, $\beta' \simeq 2.5$ and thus, $\varepsilon = 0.04-0.02$.

The solution of system (11)-(13) along with boundary conditions 1)-4) can be obtained by numerical methods. But the analytic solution of the problem is of great interest, since it would allow us to estimate the effect of the parameters on the velocity distribution and on the temperature fields in the deformation region.

The dependence of the viscosity of the material on the temperature is described by Eq. (14). For thermoplastics the coefficient b is determined experimentally and has the order 0.1-0.03 [1], and so we limit ourselves to the linear term in the expansion of the exponent

$$\mu(T) = \mu_0 \exp\{-b(T - T_0)\} \simeq \mu_0 (1 - \delta\theta), \tag{14}$$

where θ is the relative temperature: $\theta = (T - T_0)/T_0$. We expand both velocity components, the pressure gradient, and the relative temperature in powers of the small parameter δ,

$$u(\alpha, \beta) = u_0(\alpha, \beta) + \delta u_1(\alpha, \beta) + \dots, \tag{15}$$

$$v(\alpha, \beta) = v_0(\alpha, \beta) + \delta v_1(\alpha, \beta) + \dots, \tag{16}$$

$$\frac{\partial}{\partial\beta} P = \frac{\partial}{\partial\beta} P_0 + \delta \frac{\partial}{\partial\beta} P_1 + \dots, \tag{17}$$

$$\theta(\alpha, \beta) = \theta_0(\alpha, \beta) + \delta\theta_1(\alpha, \beta) + \dots \tag{18}$$

After substituting (15)-(18) into the initial system of equations (11)-(13) and setting equal the coefficients having identical powers of δ, we obtain the following system of equations for δ in the zeroth power:

$$\frac{\partial}{\partial\beta}(hu_0) + \frac{\partial}{\partial\alpha}(hv_0) = 0; \tag{19}$$

$$\frac{\partial}{\partial\beta} P_0 = \frac{\partial}{\partial\alpha}\left\{\mu_0 \left|\frac{\partial}{\partial\alpha}\left(\frac{u_0}{h}\right)\right|^n \operatorname{sign}\left[\frac{\partial}{\partial\alpha}\left(\frac{u_0}{h}\right)\right]\right\}; \quad \frac{\partial}{\partial\alpha} P_0 = 0; \tag{20}$$

$$\frac{\lambda T_0}{h^2}\theta_{0\alpha\alpha}'' - \frac{\rho c_v T_0}{h}(u_0\theta_{0\beta}' + v_0\theta_{0\alpha}') + \mu_0 \left|\frac{\partial}{\partial\alpha}\left(\frac{u_0}{h}\right)\right|^{n+1} = 0. \tag{21}$$

A system similar to (19)-(21) is also obtained for δ in the first power. In system (19)-(21) the equation of motion is independent of the energy equation, i.e., in the first approximation the problem is isothermic. Thus, by solving Eqs. (19) and (20) together we determine u_0 and v_0, and we find θ_0 from Eq. (21). Then θ_0 is substituted into the equation of motion for δ in the first power, and u_1 and v_1 are determined. The procedure is repeated until the given accuracy is reached.

We integrate Eq. (19) over α, and taking boundary condition (4) into account, we obtain

$$Q = \int_0^{\alpha_0} (hu_0)\, d\alpha = \text{const} = Vh^*\alpha_0, \tag{22}$$

$$h^* = \int_0^{\alpha_0} \frac{a}{\operatorname{ch}\alpha - \cos\beta}\, d\alpha = \frac{2a}{\alpha_0 \sin(\beta_-)} \operatorname{arctg}\left[\operatorname{th}\frac{\alpha_0}{2} \operatorname{ctg}\left(\frac{\beta_-}{2}\right)\right]. \tag{23}$$

To determine $\partial P_0/\partial\beta$, we integrate Eq. (20) twice over α, and, by solving it together with (22), we obtain

$$V\alpha_0 h^* = \int_0^{\alpha_0} Vh\, d\alpha - \int_0^{\alpha_0} h^2 \int_\alpha^{\alpha_0} \alpha^{m+1} \left|\frac{1}{\mu_0} \cdot \frac{\partial}{\partial\beta} P_0\right| \operatorname{sign}\left[\frac{\partial}{\partial\beta} P_0\right] d\alpha d\alpha. \tag{24}$$

Equation (24) can be considerably simplified if we expand h in powers of $\cosh\alpha - 1/1 - \cos\beta$ and take as the first term of the expansion

$$h = \frac{a}{1 - \cos\beta}\left(1 - \frac{\operatorname{ch}\alpha - 1}{1 - \cos\beta} + \dots\right);$$

thus,

$$\frac{\alpha_0 V(h^* - h)}{h^2} = \int_0^{\alpha_0} \int_\alpha^{\alpha_0} \alpha^m \left|\frac{1}{\mu_0} \cdot \frac{\partial}{\partial\beta} P_0\right| \operatorname{sign}\left[\frac{\partial}{\partial\beta} P_0\right] d\alpha d\alpha. \tag{25}$$

After we integrate (25) by parts, since the sign of $\partial P_0/\partial\beta$ agrees with the sign $(h^* - h)$, we obtain the pressure distribution in the deformation region,

$$P_0(\beta) = \frac{\mu_0 V(m+2)}{\alpha_0^{m+1}} \int\limits_{\beta_+}^{\beta} \left| \frac{h - h^*}{h^2} \right|^n \operatorname{sign}(h - h^*)\, d\beta; \tag{26}$$

thus, both velocity components are

$$u_0 = V\left(1 - \frac{(h - h^*)(m+2)(\alpha^{m+1} - \alpha_0^{m+1})}{h(m+1)\alpha_0^{m+1}}\right), \tag{27}$$

$$v_0 = -V \frac{\alpha(\alpha^{m+1} - \alpha_0^{m+1})h'_\beta}{(m+1)\alpha_0^{m+1} h}. \tag{28}$$

We study the solution of energy equation (21). We can isolate two zones [3] for the flow in the canals of viscous materials with low thermal conductivity. In the first zone we exclude the region directly adjoining the roller, and so the energy balance is maintained in the basic convective heat transfer along the flow and the dissipation, since the estimation of the Péclet number shows that it has the order $10^2 - 10^3$ in this zone [4]. In the second zone, directly adjoining the roller, the temperature gradients are considerable and the thermal conductivity plays a significant role. Thus, we seek the solution of Eq. (21) as a sum of the solutions of equations that describe the processes in the first and second zones.

We find the solution of the equation for the first zone:

$$\frac{\rho c_v T_0}{h} \tilde{u}_0 \frac{\partial \theta_0^{\mathrm{I}}}{\partial \beta} = \mu_0 \left| \frac{\partial}{\partial \alpha}\left(\frac{\tilde{u}_0}{h}\right)\right|^{n+1}. \tag{29}$$

We integrate over β, and we find the temperature distribution in the deformation region where for $\alpha = \alpha_0$, $\theta_0^{\mathrm{I}} = \theta_0^{\mathrm{I}}(\alpha_0, \beta)$. In order that boundary condition (1) be satisfied (for example, for the case $T_1 = T_0 = T_2$), the solution of the equation for the second zone for $\alpha = \alpha_0$ must have the form $\theta_0^{\mathrm{II}} = -\theta_0^{\mathrm{I}}$. Thus, the general solution is $\theta_0 = \theta_0^{\mathrm{I}} + \theta_0^{\mathrm{II}}$. Near the roller surface $u_0 \simeq V$, so we find v_0 from the equation of continuity,

$$v_0 = -\frac{1}{h}\int\limits_{\alpha}^{\alpha_0} (u_0 h)'_\beta\, d\alpha = -\frac{V}{h} h'_\beta(\alpha_0 - \alpha).$$

We determine the mean velocity $\tilde{u}$ as follows:

$$\tilde{u} h \alpha_0 = V \alpha_0 h^* \quad \text{or} \quad \tilde{u} = V h^*/h$$

and introduce the new variables (ξ, η):

$$\xi = \beta/\beta', \quad \eta = \sigma(\alpha_0 - \alpha), \qquad \sigma = \sqrt{\frac{\lambda\beta'}{V\rho c_v h^*}} = \left(\frac{\beta'}{\mathrm{Pe}}\right)^{\frac{1}{2}}. \tag{30}$$

For polymer materials the dimensionless value of σ is of the order of 10^{-2}. Thus, the homogeneous part of Eq. (21), which describes the temperature distribution in the second zone, takes the form

$$\frac{\partial^2 \theta_0^{\mathrm{II}}}{\partial \eta^2} - \frac{h'_\xi}{h^*} \eta \cdot \frac{\partial \theta_0^{\mathrm{II}}}{\partial \eta} - \frac{\partial \theta_0^{\mathrm{II}}}{\partial \xi} = 0. \tag{31}$$

We replace the variables once more in order to release the term $[\theta_0^{\mathrm{II}}]'_\eta$:

$$p = \eta \exp\{-h/h^*\}, \quad q = \int\limits_{\xi_+}^{\xi} \exp\left\{-\frac{2h}{h^*}\right\} d\xi. \tag{32}$$

Thus, Eq. (31) takes the form

$$\frac{\partial^2}{\partial p^2}\theta_0^{\mathrm{II}} - \frac{\partial}{\partial q}\theta_0^{\mathrm{II}} = 0, \tag{33}$$

and the boundary conditions take the form

$$\theta_0^{\mathrm{II}} = -\theta_0^{\mathrm{I}} \ \text{for} \ p = 0, \text{ i.e., } \alpha = \pm\alpha_0,$$
$$\theta_0^{\mathrm{II}} = 0 \ \text{for} \ q = 0.$$

To solve Eq. (33), we use the operational method of [8]. By transformations we obtain an ordinary differential equation with respect to p,

$$s\bar{\theta}_0^{\mathrm{II}} = \frac{d^2}{dp^2}\bar{\theta}_0^{\mathrm{II}}; \quad \bar{\theta}_0^{\mathrm{II}}(s, p)|_{p=0} = L(-\theta_0^{\mathrm{I}}(p, q)).$$

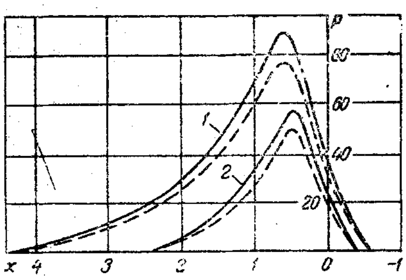

Fig. 2

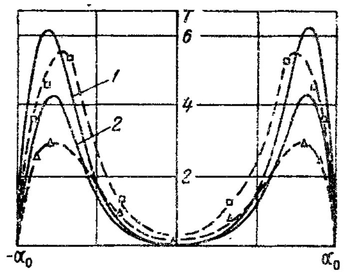

Fig. 3

Fig. 2. Distribution of a specific pressure in the deformation region for different values of β_+: V = 10.48 cm/sec; $2h_0$ = 0.1 cm; R = 8 cm; n = 0.2; μ = 1.67 kg · sec^n/cm^2; dashed curve, experiment; solid curve, calculation. p, kgf/cm^2; x, cm.

Fig. 3. Temperature field in the cross section of a minimum gap: V = 15.6 cm/sec; $2h_0$ = 0.1 cm; R = 8 cm. For curve 1: n = 0.2; μ_0 = 1.67 kg · sec^n/cm^2; λ = 0.136 kcal/m · h · °C; for curve 2: n = 0.224; μ_0 = 1.63 kg · sec^n/cm^2; λ = 0.205 kcal/m · h · °C. Points correspond to the experiment. T, °C,

The general solution of this equation is

$$\bar{\theta}_0^{11} = c_1 \exp(-\sqrt{s}\,p) + c_2 \exp(\sqrt{s}\,p);$$

$c_2 = 0$, since in the opposite case the solution increases as $p \to \infty$. Consequently,

$$\bar{\theta}_0^{11}(p,\ q) = L^{-1}(\bar{\theta}_0^{11}(s,\ 0)\exp(-\sqrt{s}\,p)).$$

We find the original according to the tables, and, using a Duhamel integral, we write the solution

$$\theta_0^{11}(p,\ q) = \int_0^q \theta_0^{1'}(q-z)\ \mathrm{Erf}\left(\frac{p}{2\sqrt{q}}\right) dz. \tag{34}$$

In Figs. 2 and 3 we represent the distribution of the specific pressure and temperature in the deformation region in a 160 × 320-mm laboratory calender for two types of polymer materials. In calculating the temperature field and the pressure distribution in the deformation region as applied to the processing of polymers on calenders, we can consider the following. It is known that when there is a large reserve of material in the deformation region of the rollers a so-called "stagnation zone" ("rotating reserve") is formed, in which the reprocessed material is in circulatory motion until it is drawn into the region where it moves in the direction of the roller rotation. In solving the given problem, we do not intend to describe the trajectories of the motion of the material in the "rotating reserve."

The flow process in the zone where $v_x \geq v_y$ (regions A and B in Fig. 1) has a great effect on the temperature distribution in the deformation region, the specific pressures, and the thrust forces. The calculation method presented above allows us to determine the flow characteristics in this zone. It is clear that boundary condition 1) is approximate for negative velocities. However, we see from Fig. 2 that the pressure distribution satisfactorily agrees with the experimental data both for small (curve 2), as well as for large (curve 1), reserves of the material.

Thus, the suggested mathematical model satisfactorily describes the flow of a pseudoplastic material between two rotating cylinders. The calculations show that for computing the pressure distribution, thrust forces, and temperature fields in the deformation region, sufficient accuracy for engineering practice is guaranteed by the system of equations (19)-(21) for δ in the zeroth power.

NOTATION

τ^{ij}, stress tensor; e^{ij}, strain rate tensor; J_2, second invariant of tensor e^{ij}; h, metric tensor; $2a = 2\sqrt{2Rh_0}$, distance between poles; v, projection of velocity on axis α; u, projection of velocity on axis β; T, temperature; P, pressure; n, μ, rheological constants; μ_0, viscosity at $T = T_0$; $\alpha_0 = \ln[(R + h_0)/(R - h_0)]$, coordinate of roller surface; β', change of β coordinate from material entrance into deformation region and to its outlet; $\beta_+ = 2\arctan(x_+/a)$ entrance coordinate; β_-, outlet coordinate; ρ, density; λ, thermal conductivity; c_V, heat capacity; T_1, T_2, temperatures of roller surfaces; 2Q, material discharge; V, speed of roller surface rotation.

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INFLUENCE OF GRAVITATIONAL CONVECTION ON THE PROGRESS OF A HETEROGENEOUS CATALYTIC REACTION UNDER ISOTHERMAL CONDITIONS

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The critical conditions for origination of natural gravitational convection during the progress of a heterogeneous catalytic reaction are considered. The influence of developed convection on the reaction progress under isothermal conditions is analyzed.

As is known, the macroscopic velocity of a heterogeneous catalytic reaction depends on the relationship between the true reaction rate constant and the intensity of mass transfer [1]. The intensity of mass transfer evidently increases in the presence of gravitational convection. This can result in the passage from one mode of reaction progress to another. In other words, [if the reaction were to proceed in the diffusion domain without natural convection and the rate of mass transfer were limited] then the reaction rate can set the limiting stage for sufficiently strong convection.

This paper is devoted to a clarification of the role of natural gravitational convection in the progress of a heterogeneous catalytic reaction. However, the solution of this question requires knowledge of the conditions for origination of gravitational convection due to the progress of a heterogeneous catalytic reaction.

1. Critical Conditions for Origination of Convection

Let us consider an infinite plane horizontal layer filled with fluid or gas and bounded by solid boundaries. The temperatures on the boundaries are identical and do not vary with time. A catalytic reaction of the type

$$\nu_1 A_1 \xrightarrow{K} A_2 \quad (1)$$

proceeds on the upper boundary of the layer, where A_1 is the provisional notation for the initial material, A_2

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